PP: 9-14

TRINITY COLLEGE

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNIT 3 and 4

Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

SOLUTIONS

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: five minutes

Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet

Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	48	35
Section Two: Calculator-assumed	13	13	100	101	65
			Total	149	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

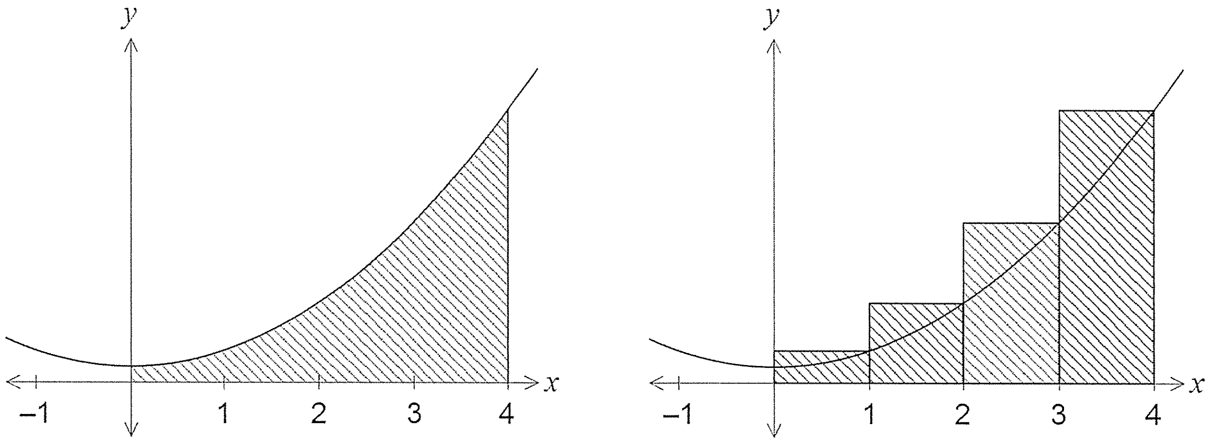
Section One: Calculator-free **35% (48 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(5 marks)**

Part of the graph of $y = x^2 + 1$ is shown in the diagrams below.



An approximation for the area beneath the curve between $x = 0$ and $x = 4$ is made using rectangles as shown in the right-hand diagram. Determine the exact amount by which the approximate area exceeds the exact area.

$$A = \int_0^4 (x^2+1)\,dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^4$$

$$= \frac{64}{3} + 4$$

$$= \frac{76}{3}$$

$$A = (1\times2)+(1\times5)+(1\times10)+(1\times17)$$

$$= 34$$

DIFFERENCE $= \frac{26}{3}$

Question 2 **(9 marks)**

(a) Differentiate the following with respect to x. Do not simplify your answer.

(i) $y = \int_x^1 (t - t^3)\, dt$. (2 marks)

$y' = -\frac{d}{dx} \int_1^x (t - t^3)\, dt = x^3 - x$

(ii) $y = \dfrac{\sin(2x+1)}{e^{-x}}$. (3 marks)

$y' = \dfrac{e^{-x} \cos(2x+1)2 + e^{-x} \sin(2x+1)}{e^{-2x}}$

(b) Determine the values of the constants a, b and c, given that $f''(x) = e^{3x}\left(ax^2 + bx + c\right)$ when $f(x) = x^2 e^{3x}$. (4 marks)

$f'(x) = 2xe^{3x} + 3x^2 e^{3x}$

$f''(x) = 2e^{3x} + 6xe^{3x} + 6xe^{3x} + 9x^2 e^{3x}$

$= 2e^{3x} + 12xe^{3x} + 9x^2 e^{3x}$

$= e^{3x}\left(2 + 12x + 9x^2\right)$

$a = 9$

$b = 12$

$c = 2$

Question 3 **(6 marks)**

A function $P(x)$ is such that $\dfrac{dP}{dx} = ax^2 - 12x$, where a is a constant and the graph of $y = P(x)$ has a stationary point at (4, 8). Determine $P(10)$.

$P'(4) = 0 \quad \therefore \quad 16a - 48 = 0$

$a = 3$

$P'(x) = 3x^2 - 12x$

$P(x) = x^3 - 6x^2 + c$

$P(4) = 8 \quad \therefore \quad 64 - 96 + c = 8$

$c = 40$

$P(x) = x^3 - 6x^2 + 40$

$P(10) = 1000 - 600 + 40$

$= 440$

Question 4 **(7 marks)**

Consider the function defined by $f(x) = \dfrac{x}{2} - \sqrt{x}$, $x \geq 0$.

(a) Determine the coordinates of the stationary point of $f(x)$. (3 marks)

$$f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2} - \frac{1}{2\sqrt{x}} = 0$$

$$\frac{1}{2} = \frac{1}{2\sqrt{x}}$$

$$x = 1$$

$$f(1) = \frac{1}{2} - \sqrt{1}$$

$$= -\frac{1}{2}$$

STATIONARY PT $\left(1, -\frac{1}{2}\right)$

(b) Use the second derivative test to determine the nature of the stationary point found in (a). (3 marks)

$$f''(x) = \frac{1}{4\sqrt{x}^3}$$

$$f''(1) = +ve$$

$\therefore$ LOCAL MIN

(c) State the global minimum of $f(x)$. (1 mark)

$$-\frac{1}{2}$$

Question 5 **(5 marks)**

The area of a segment with central angle θ in a circle of radius r is given by $A = \frac{r^2}{2}(\theta - \sin\theta)$.

Use the increments formula to approximate the increase in area of a segment in a circle of radius 10 cm as the central angle increases from $\frac{\pi}{3}$ to $\frac{11\pi}{30}$.

$r = 10$ $\quad A = 50(\theta - \sin\theta)$

$$\frac{dA}{d\theta} = 50(1 - \cos\theta)$$

$\theta = \frac{\pi}{3}$ $\quad \frac{dA}{d\theta} = 50\left(1 - \cos\frac{\pi}{3}\right)$

$$= 50\left(1 - \frac{1}{2}\right)$$

$$= 25$$

$$\delta\theta = \frac{11\pi}{30} - \frac{10\pi}{30}$$

$$= \frac{\pi}{30}$$

$$\delta A \approx \frac{dA}{d\theta} \times \delta\theta$$

$$\approx 25 \times \frac{\pi}{30}$$

$$= \frac{5\pi}{6} \text{ cm}^2$$

Question 6 **(5 marks)**

(a) Differentiate $y = \dfrac{2x+1}{e^x}$, simplifying your answer. (3 marks)

$$y' = \frac{2e^x - (2x+1)e^x}{e^{2x}}$$

$$= \frac{2e^x - 2xe^x - e^x}{e^{2x}}$$

$$= \frac{e^x - 2xe^x}{e^{2x}}$$

$$= \frac{e^x(1-2x)}{e^{2x}} = \frac{1-2x}{e^x}$$

(b) Evaluate $\int_1^2 \left(\dfrac{1-2x}{e^x}\right) dx$. (2 marks)

$$= \left[\frac{2x+1}{e^x}\right]_1^2$$

$$= \frac{5}{e^2} - \frac{3}{e}$$

$$\left(= \frac{5-3e}{e^2}\right)$$

Question 7 **(6 marks)**

The discrete random variable X has the probability distribution shown in the table below.

x	0	1	2	3
$P(X=x)$	$\frac{2a^2}{3}$	$\frac{1-3a}{3}$	$\frac{1+2a}{3}$	$\frac{4a^2}{3}$

Determine the value of the constant a.

$$\frac{2a^2 + 1-3a + 1+2a + 4a^2}{3} = 1$$

$$6a^2 - a - 1 = 0$$

$$(3a+1)(2a-1) = 0$$

$$a = -\frac{1}{3}, \frac{1}{2}$$

$a = \frac{1}{2}$ $\left[\frac{1}{6}, -\frac{1}{6}, \frac{2}{3}, \frac{1}{3}\right]$

-ve PROBABILITY

$\therefore a \neq \frac{1}{2}$

$a = -\frac{1}{3}$

Question 8 **(5 marks)**

The area bounded by the curve $y = e^{2-x}$ and the lines $y = 0$, $x = 1$ and $x = k$ is exactly $e - 1$ square units. Determine the value of the constant k, given that $k > 1$.

$$\int_1^k e^{2-x}\,dx = -\int_1^k -e^{2-x}\,dx$$

$$= -\left[e^{2-x}\right]_1^k$$

$$= -\left(e^{2-k} - e\right)$$

$$= e - e^{2-k}$$

$$e - e^{2-k} = e - 1$$

$$e^{2-k} = 1$$

$$\therefore\ 2 - k = 0$$

$$k = 2$$

End of questions

TRINITY COLLEGE

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNITS 3 and 4

Section Two: Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work:	ten minutes
Working time for section:	one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet

Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items:	pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items:	drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section Two: Calculator-assumed **65% (101 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(5 marks)**

A recent news report said that it took 34 months for the population of Australia to increase from 23 to 24 million people.

(a) Assuming that the rate of growth of the population can be modelled by the equation $\frac{dP}{dt} = kP$, where P is the population of Australia at time t months, determine the value of the constant k. (3 marks)

$P = P_0 e^{kt}$

$24 = 23 e^{34t}$

$k = 0.00125$

(b) Assuming the current rate of growth continues, how long will it take for the population to increase from 24 million to 25 million people? (2 marks)

$25 = 24 e^{0.00125t}$

$t = 32.6$ MONTHS

Question 10 **(8 marks)**

The discrete random variable Y has the probability distribution shown in the table below.

y	-2	-1	0	1	2
$P(Y=y)$	0.4	0.2	0.1	0.1	0.2

(a) Determine $P(Y \geq 0 \mid Y \leq 1)$. (2 marks)

$= \frac{0.2}{0.8}$

$= \frac{1}{4}$

(b) Calculate

(i) $E(Y)$. (2 marks)

$= (-2)(0.4) + (-1)(0.2) + 0(0.1) + 1(0.1) + 2(0.2)$

$= -0.5$

(ii) $E(1-2Y)$. (1 mark)

$= 1 - 2(-0.5)$

$= 2$

(c) Calculate

(i) $Var(Y)$. (2 marks)

$= 0.4(-1.5)^2 + (0.2)(-0.5)^2 + (0.1)(0.5)^2 + (0.1)(1.5)^2 + (0.2)(2.5)^2$

$= 2.45$

(ii) $Var(1-2Y)$. (1 mark)

$= (-2)^2 \, VAR(Y)$

$= 9.8$

$SD(Y) = \sqrt{2.45}$

$SD(1-2Y) = -2\sqrt{2.45}$

$VAR(1-2Y) = (-2\sqrt{2.45})^2$

$= 9.8$

Question 11 **(7 marks)**

It is known that 15% of Year 12 students in a large country study advanced mathematics.

A random sample of n students is selected from all Year 12s in this country, and the random variable X is the number of those in the sample who study advanced mathematics.

(a) Describe the distribution of X. (2 marks)

$X \sim BIN(n, 0.15)$

(b) If $n = 22$, determine the probability that

(i) three of the students in the sample study advanced mathematics. (1 mark)

$P(X=3) = 0.2370$

(ii) more than three of the students in the sample study advanced mathematics. (1 mark)

$P(X \geq 4) = 0.4248$

(iii) none of the students in the sample study advanced mathematics. (1 mark)

$P(X=0) = 0.0280$

(c) If ten random samples of 22 students are selected, determine the probability that at least one of these samples has no students who study advanced mathematics. (2 marks)

$Y \sim BIN(10, 0.028)$

$P(Y \geq 1) = 0.247$

Question 12 **(8 marks)**

The height of grain in a silo, initially 0.4 m, is increasing at a rate given by $h'(t) = 0.55t - 0.05t^2$ for $0 \le t \le 11$, where h is the height of grain in metres and t is in hours.

(a) At what time is the height of grain rising the fastest? (2 marks)

$h''(t) = 0.55 - 0.1t$

$0.55 - 0.1t = 0$

$t = 5.5$ HRS

(b) Determine the height of grain in the silo after 11 hours. (3 marks)

HEIGHT CHANGE $= \int_0^{11} 0.55 - 0.1t \, dt$

$= 11.09$

$h = 11.09 + 0.4$

$= 11.49$ m

(c) Calculate the time taken for the grain to reach a height of 4.45 m. (3 marks)

HEIGHT CHANGE $= 4.45 - 0.4$

$= 4.05$

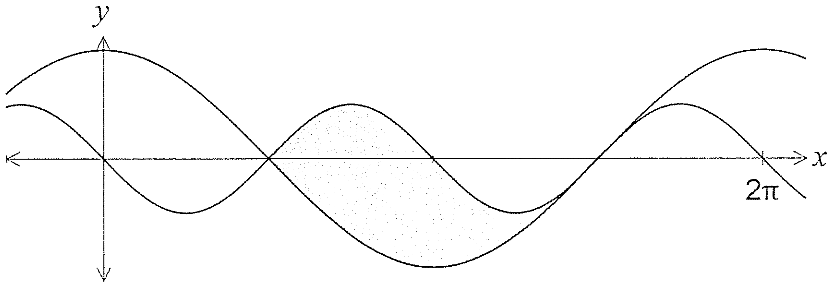
$\int_0^{k} 0.55t - 0.05t^2 \, dt = 4.05$

$\frac{11k}{40} - \frac{k^3}{60} = 4.05$

$k = 4.5$ HOURS $(0 \le t \le 11)$

Question 13 **(6 marks)**

The shaded region on the graph below is enclosed by the curves $y = -\sin(2x)$ and $y = 2\cos x$.



Show that the area of the region is 4 square units.

$2\cos x = -\sin 2x$

$2\cos x + 2\sin x \cos x = 0$

$2\cos x\,(1 + \sin x) = 0$

$\cos x = 0$ $\quad\quad$ $\sin x = -1$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad\quad$ $x = \frac{3\pi}{2}$

$$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-\sin 2x - 2\cos x)\, dx$$

$$= \left[\frac{1}{2}\cos 2x - 2\sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left(-\frac{1}{2} + 2\right) - \left(-\frac{1}{2} - 2\right)$$

$$= 4$$

Question 14 **(14 marks)**

(a) Determine the mean of a Bernoulli distribution with variance of 0.24. (3 marks)

$p(1-p) = 0.24$

$p = 0.4,\ 0.6$ (2 POSSIBLE MEANS)

(b) A Bernoulli trial, with probability of success p, is repeated n times. The resulting distribution of the number of successes has an expected value of 5.76 and a standard deviation of 1.92. Determine n and p. (4 marks)

$X \sim BIN(n, p)$

$np = 5.76$

$np(1-p) = 1.92^2$

$5.76(1-p) = 1.92^2$

$p = 0.36$

$n = 16$

(c) The probability that a student misses their bus to school is 0.2, and the probability that they miss the bus on any day is independent of whether they missed it on the previous day.

Over five consecutive weekdays, what is the probability that the student

(i) only misses the bus on Tuesday? (2 marks)

$0.2 \times 0.8^4 = 0.0819$

(ii) misses the bus at least twice? (2 marks)

$X \sim BIN(5, 0.2)$

$P(X \geq 2) = 0.2627$

(iii) misses the bus on Tuesday and on two other days? (3 marks)

$Y \sim BIN(4, 0.2)$

$P = 0.2 \times P(Y = 2)$

$= 0.2 \times 0.1536$

$= 0.0307$

Question 15 **(9 marks)**

A particle moves in a straight line according to the function $x(t) = \frac{t^2+3}{t+1}$, $t \geq 0$, where t is in seconds and x is the displacement of the particle from a fixed point O, in metres.

(a) Determine the velocity function, $v(t)$, for the particle. (2 marks)

$$v(t) = \frac{d}{dt} x(t)$$

$$= \frac{t^2 + 2t - 3}{(t+1)^2}$$

(b) Determine the displacement of the particle at the instant it is stationary. (2 marks)

$v(t) = 0$ ie $\frac{t^2+2t-3}{(t+1)^2} = 0$

$$t^2 + 2t - 3 = 0$$

$$(t+3)(t-1) = 0$$

$$t = 1, \cancel{-3}$$

$$x(1) = 2 \text{ m}$$

(c) Show that the acceleration of the particle is always positive. (2 marks)

$$a(t) = \frac{d}{dt} v(t)$$

$$= \frac{8}{(t+1)^3}$$

SINCE $t \geqslant 0$ $a(t) > 0$

(d) After five seconds, the particle has moved a distance of k metres.

(i) Explain why $k \neq \int_0^5 v(t)\,dt$. (1 mark)

$\int_0^5 v(t)\,dt$ GIVES CHANGE IN DISPLACEMENT

(ii) Calculate k. (2 marks)

$$k = \int_0^5 |v(t)|\,dt$$

$$= \frac{11}{3}$$

Question 16 **(7 marks)**

The loudness of a sound is measured in decibels.
The loudness scale is constructed by comparing the energy output (E) of any sound source with the energy output (E_0) of a barely audible sound source.

Loudness, $L = 10\log(\frac{E}{E_0})$ decibels.

(a) Use the Loudness equation above to determine the value of p, q and r as shown below. (3 marks)

Sound Source	Energy Output	Loudness (Decibels)
Barely audible	E_0	p
Circular Saw	q	98
Jet Engine	$10^{14.2} \times E_0$	r

$p = 0$

$q =$

$r =$

(b) Use an algebraic method to solve the equation $(\log_3 x)^2 = \log_3 x^2$ (4 marks)

$(\log_3 x)^2 - 2\log_3 x = 0$

$\log_3 x(\log_3 x - 2) = 0$

$\log_3 x = 0$

$x = 3^0$

$= 1$

$\log_3 x - 2 = 0$

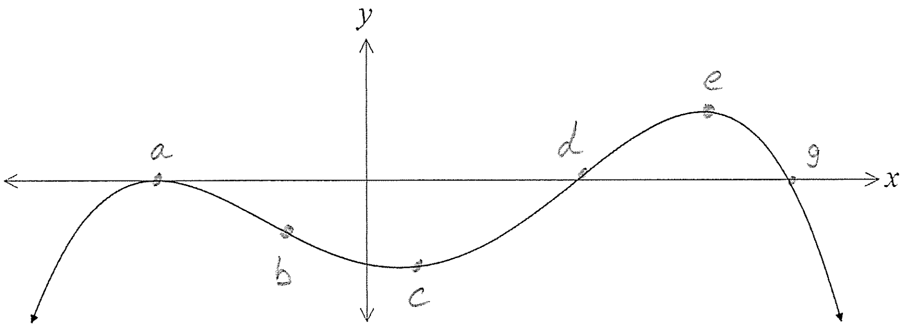
$\log_3 x = 2$

$x = 3^2$

$= 9$

Question 17 **(8 marks)**

The graph of $y = f'(x)$, the derivative of a polynomial function f, is shown below. The graph of $y = f'(x)$ has stationary points when $x = a$, $x = c$ and $x = e$, points of inflection when $x = b$ and $x = d$, and roots when $x = a$, $x = d$ and $x = g$, where $a < b < c < d < e < g$.



(a) For what value(s) of x does the graph of $y = f(x)$ have a point of inflection? (1 mark)

a, c, e

(b) Does the graph of $y = f(x)$ have a local maximum? Justify your answer. (2 marks)

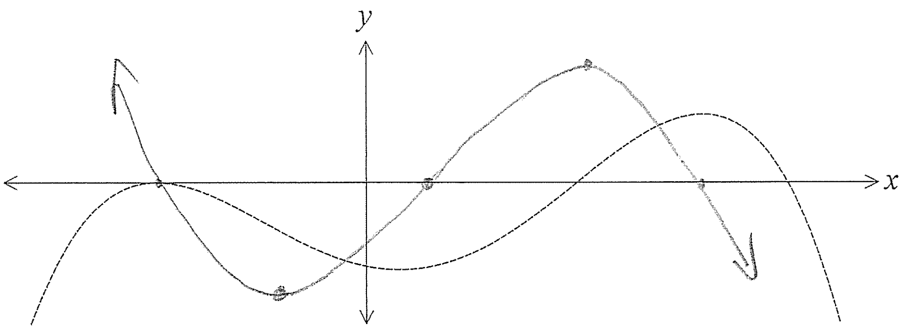
YES AT $x = g$

FOR MAX, GRADIENT MUST GO FROM POSITIVE TO ZERO TO NEGATIVE

(c) Does the graph of $y = f(x)$ have a horizontal point of inflection? Justify your answer. (2 marks)

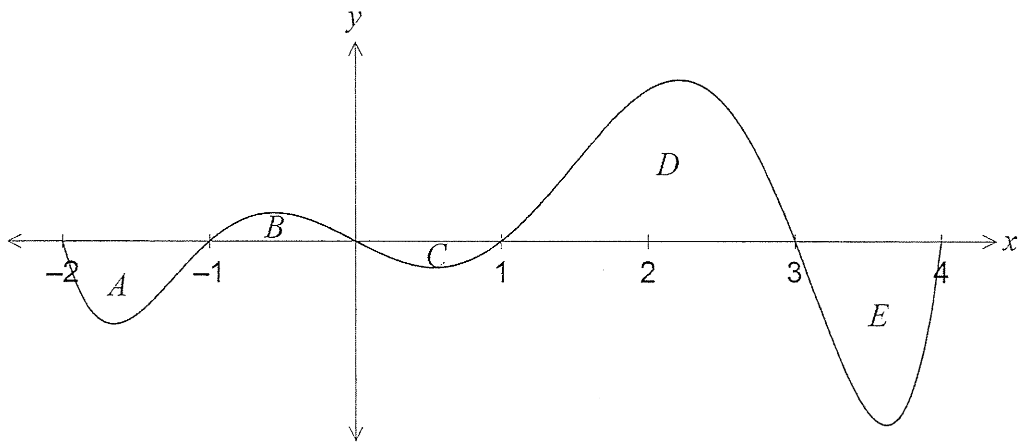
YES PT. OF INFLECTION AT a AS GRADIENT GOES FROM NEGATIVE TO ZERO TO NEGATIVE

(d) On the axis below, sketch a possible graph of $y = f''(x)$. The graph of $y = f'(x)$ is shown with a broken line for your reference. (3 marks)



Question 18 **(8 marks)**

The graph of the function $y = f(x)$ is shown below for $-2 \le x \le 4$.



The area of regions enclosed by the x-axis and the curve, A, B, C, D and E, are 12, 7, 5, 32 and 21 square units respectively.

(a) Determine the value of $\int_{-2}^{4} f(x)\,dx$. (2 marks)

$-12 + 7 - 5 + 32 - 21$

$= 1$

(b) Determine the area of the region enclosed between the graph of $y = f(x)$ and the x-axis from $x = 0$ to $x = 4$. (2 marks)

$5 + 32 + 21$

$= 58$

(c) Determine the values of

(i) $\int_{0}^{3} f(x) + 3\,dx$. $= \int_0^3 f(x)\,dx + \int_0^3 3\,dx$ (2 marks)

$= (-5 + 32) + [3x]_0^3$

$= 27 + 9$

$= 36$

(ii) $\int_{-2}^{3} \frac{f(x)}{2}\,dx$. $= \frac{1}{2}\int_{-2}^{3} f(x)\,dx$ (2 marks)

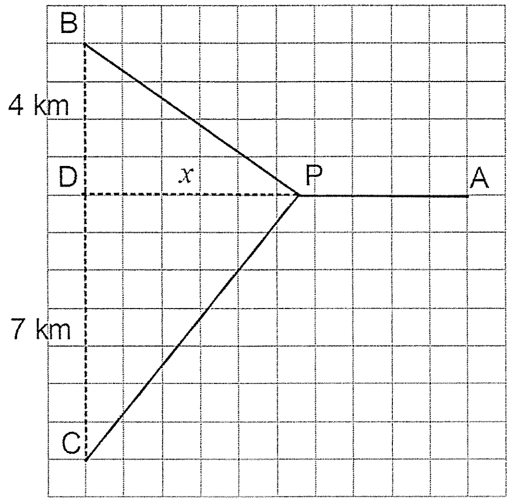
$= \frac{1}{2}(-12 + 7 - 5 + 32)$

$= 11$

Question 19 **(7 marks)**

Three telecommunication towers, A, B and C, each need underground power cable connections directly to a new power station, P, that is to be built x km from depot D on a 10 km road running east-west between D and A.

Tower B lies 4 km due north of depot D and tower C lies 7 km south of the depot, as shown in the diagram.



(a) Determine an expression for the total length of underground cable required to connect A, B and C directly to P. (2 marks)

$$L = \sqrt{16+x^2} + \sqrt{49+x^2} + 10 - x$$

(b) Show that the minimum length of cable occurs when $\dfrac{x}{\sqrt{16+x^2}} + \dfrac{x}{\sqrt{49+x^2}} = 1$. (3 marks)

$$\frac{dL}{dx} = \frac{1}{2}(2x)(16+x^2)^{-\frac{1}{2}} + \frac{1}{2}(2x)(49+x^2)^{-\frac{1}{2}} - 1$$

$$\frac{dL}{dx} = 0 \qquad \frac{x}{\sqrt{16+x^2}} + \frac{x}{\sqrt{49+x^2}} = 1$$

(c) Determine the minimum length of cable required. (2 marks)

$x = 3.0255$

$L = 19.6$ km

Question 20 **(7 marks)**

Consider the function $f(t) = \dfrac{t-4}{2}$ and the function $A(x) = \int_0^x f(t)\, dt$.

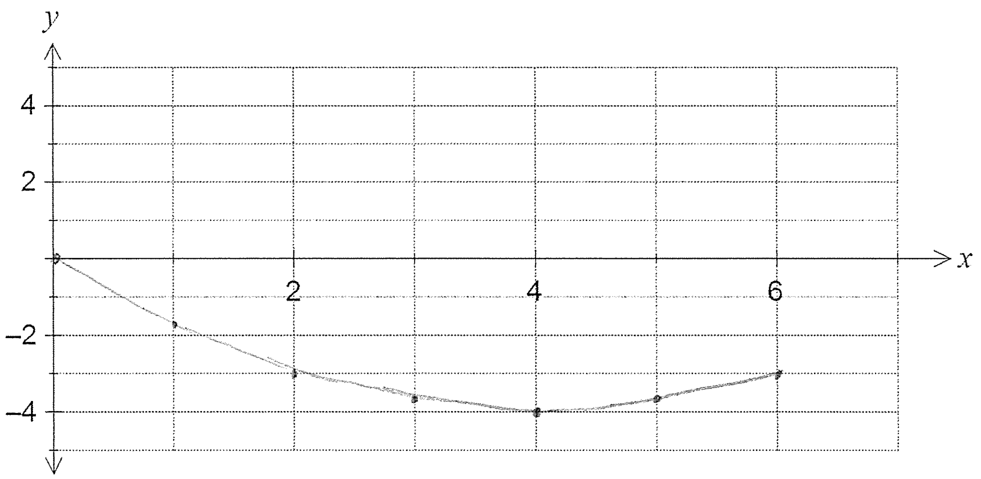
(a) Complete the table below. (2 marks)

x	0	1	2	3	4	5	6
$A(x)$	0	-1.75	−3	−3.75	−4	−3.75	−3

(b) For what value(s) of x is the function $A(x)$ increasing? (1 mark)

$x > 4$

(c) On the axes below, sketch the graph of $y = A(x)$ for $0 \le x \le 6$. (2 marks)



(d) Determine

(i) when $A'(x) = 0$. (1 mark)

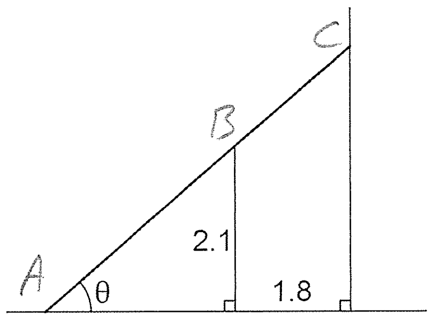
$x = 4$

(ii) the function $A(x)$ in terms of x. (1 mark)

$\int \frac{t}{2} - 2 = \frac{x^2}{4} - 2x + \cancel{c}$ $c = 0$

Question 21 **(7 marks)**

A vertical wall, 2.1 metres tall, stands on level ground and 1.8 metres away from the wall of a house. A ladder, of negligible width, leans at an angle of θ to the ground and just touches the ground, wall and house, as shown in the diagram.



(a) Show that the length of the ladder, L, is given by $L = \dfrac{2.1}{\sin\theta} + \dfrac{1.8}{\cos\theta}$. ~~(3~~ marks)

4

$\sin\theta = \dfrac{2.1}{AB}$ $\cos\theta = \dfrac{1.8}{BC}$

$AB = \dfrac{2.1}{\sin\theta}$ $BC = \dfrac{1.8}{\cos\theta}$

$L = AB + BC$

$= \dfrac{2.1}{\sin\theta} + \dfrac{1.8}{\cos\theta}$

(b) Determine the length of the shortest ladder that can touch the ground, wall and house at the same time. Justify your answer. ~~(4~~ marks)

3

$$\frac{dL}{d\theta} = \frac{1.8\sin^3\theta - 2.1\cos^3\theta}{\sin^2\theta\cos^2\theta}$$

$\dfrac{dL}{d\theta} = 0$ $\theta = 0.8111$

$L = 5.51$ m

End of questions